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Research Article

Fixed Point Theorems for a Weaker Meir-Keeler Type ψ-Set Contraction in Metric Spaces

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We define a weaker Meir-Keeler type function and establish the fixed point theorems for a weaker Meir-Keeler type ψ-set contraction in metric spaces.

1. Introduction and Preliminarie

In 1929, Knaster et al. [1] had proved the well-known KKM theorem on n-simplex. Besides, in 1961, Fan [2] had generalized the KKM theorem to an infinite dimensional topological vector space. Later, Amini et al. [3] had introduced the class of KKM-type mappings on metric spaces and established some fixed point theorems for this class. In this paper, we define a weaker Meir-Keeler type function and establish the fixed point theorems for a weaker Meir-Keeler type ψ-set contraction in metric spaces.

Throughout this paper, by $\Re_+$ we denote the set of all real nonnegative numbers, while $\mathbb{N}$ is the set of all natural numbers. We digress briefly to list some notations and review some definitions. Let X and Y be two Hausdorff topological spaces, and let $T : X \to 2^Y$ be a set-valued mapping. Then T is said to be closed if its graph $\mathcal{G}_T = \{(x,y) \in X \times Y : y \in T(x)\}$ is closed. T is said to be compact if the image $T(X)$ of X under T is contained in a compact subset of Y. If D is a nonempty subset of X, then $\langle D \rangle$ denotes the class of all nonempty finite subsets of D. And, the following notations are used:

(i) $T(x) = \{y \in Y : y \in T(x)\}$,

(ii) $T(A) = \cup_{x \in A} T(x)$,

(iii) $T^{-1}(y) = \{x \in X : y \in T(x)\}$, and

(iv) $T^{-1}(B) = \{x \in X : T(x) \cap B \neq \phi\}$.

Let (M,d) be a metric space, $X \subset M$ and $\delta > 0$. Let $B_M(X,\delta) = \{x \in M : d(x,X) \leqslant \delta\}$, and let $N_M(X,\delta) = \{x \in M : d(x,X) < \delta\}$.

Suppose that X is a bounded subset of a metric space (M,d). Then we define the following

(i) $co(X) = \cap\{B \subset M : B$ is a closed ball in M such that $X \subset B\}$, and

(ii) X is said to be subadmissible [3], if for each $A \in \langle X \rangle$, $co(A) \subset X$.

In 1996, Chang and Yen [4] introduced the family $\mathrm{KKM}(X,Y)$ on the topological vector spaces and got results about fixed point theorems, coincidence theorems, and its applications on this family. Later, Amini et al. [3] introduced the following concept of the $\mathrm{KKM}(X,Y)$ property on a subadmissible subset of a metric space (M,d).

Let X be an nonempty subadmissible subset of a metric space (M,d), and let Y a topological space. If $T,F : X \to 2^Y$ are two set-valued mappings such that for any $A \in \langle X \rangle$, $T(co(A)) \subset F(A)$, then F is called a generalized KKM mapping with respect to T. If the set-valued mapping $T : X \to 2^Y$ satisfies the requirement that for any generalized KKM mapping F with respest to T, the family $\{\overline{F(x)} : x \in X\}$ has finite intersection property, then T is said to have the KKM property. The class $\mathrm{KKM}(X,Y)$ is denoted to be the set $\{T : X \to 2^Y : T$ has the KKM property$\}$.

Recall the notion of the Meir-Keeler type function. A function $\psi : \mathfrak{R}_+ \to \mathfrak{R}_+$ is said to be a Meir-Keeler type function (see [5]), if for each $\eta \in \mathfrak{R}_+$, there exists $\delta = \delta(\eta) > 0$ such that for $t \in \mathfrak{R}_+$ with $\eta \le t < \eta + \delta$, we have $\psi(t) < \eta$.

We now define a new weaker Meir-Keeler type function as follows.

Definition 1.1. We call $\psi : \mathfrak{R}_+ \to \mathfrak{R}_+$ a weaker Meir-Keeler type function, if for each $\eta > 0$, there exists $\delta > 0$ such that for $t \in \mathfrak{R}_+$ with $\eta \le t < \eta + \delta$, and there exists $n_0 \in \mathbb{N}$ such that $\psi^{n_0}(t) < \eta$.

A function $\psi : \mathfrak{R}_+ \to \mathfrak{R}_+$ is said to be upper semicontinuous, if for each $t_0 \in \mathfrak{R}_+$, $\lim_{t\to t_0}\sup \psi(t) \le \psi(t_0)$. Recall also that $\psi : \mathfrak{R}_+ \to \mathfrak{R}_+$ is said to be a comparison function (see [6]) if it is increasing and $\lim_{n\to\infty}\psi^n(t) = 0$. As a consequence, we also have that for each $t > 0$, $\psi(t) < t$, and $\psi(0) = 0$, ψ is continuous at 0. We generalize the comparison function to be the other form, as follows.

Definition 1.2. We call $\psi : \mathfrak{R}_+ \to \mathfrak{R}_+$ a generalized comparison function, if ψ is upper semicontinuous with $\psi(0) = 0$ and $\psi(t) < t$ for all $t > 0$.

Proposition 1.3. *If $\psi : \mathfrak{R}_+ \to \mathfrak{R}_+$ is a generalized comparison function, then there exists a strictly increasing, continuous function $\alpha : \mathfrak{R}_+ \to \mathfrak{R}_+$ such that $\psi(t) \le \alpha(t) < t$, for all $t > 0$.*

Proof. Let $\phi(t) = t - \psi(t)$. Since $\psi : \mathfrak{R}_+ \to \mathfrak{R}_+$ is an upper semicontinuous function, hence it attains its minimum in any closed bounded interval of $\mathfrak{R}_+$.

For each $n \in \mathbb{N}$, we first define four sequences $\{a_n\}, \{b_n\}, \{c_n\}$, and $\{d_n\}$ as follows:

(i) $a_n = \min_{t\in[n,n+1]}\phi(t)$,

(ii) $b_n = \min_{t\in[1/(n+1),1/n]}\phi(t)$,

(iii) $c_1, d_1 = \min\{a_1, b_1\}$,

(iv) $c_n = \min\{c_1, a_1, a_2, \dots, a_n\}$ for $n \ge 2$, and

(v) $d_n = \min\{c_1, b_1, b_2, \dots, b_n, 1/n(n+1)\}$ for $n \ge 2$.

And, we next let a function $\alpha : \Re_+ \longrightarrow \Re_+$ satisfy the following:

(1) $\alpha(0) = 0,\ \alpha(n) = n - c_n,\ \alpha(1/n) = 1/n - d_n,$

(2) if $n \le t \le n+1$, then

$$\alpha(t) = (t-n)\alpha(n+1) + (n+1-t)\alpha(n), \tag{1.1}$$

(3) if $1/(n+1) \le t \le 1/n$, then

$$\alpha(t) = \alpha\left(\frac{1}{n+1}\right) + n(n+1)\left[\alpha\left(\frac{1}{n}\right) - \alpha\left(\frac{1}{n+1}\right)\right]\left(t - \frac{1}{n+1}\right). \tag{1.2}$$

Then by the definition of the function α, we are easy to conclude that α is strictly increasing, continuous. We complete the proof by showing that $\psi(t) \le \alpha(t)$ for all $t > 0$.

If $n \le t \le n+1$, then

$$\begin{aligned} \alpha(t) &= (t-n)\alpha(n+1) + (n+1-t)\alpha(n) \\ &= (t-c_n) + (t-n)(c_n - c_{n+1}) \\ &\ge t - \left[t - \psi(t)\right] + (t-n)(c_n - c_{n+1}) \\ &\ge \psi(t). \end{aligned} \tag{1.3}$$

If $1/(n+1) \le t \le 1/n$, then

$$\begin{aligned} \alpha(t) &= \alpha\left(\frac{1}{n+1}\right) + n(n+1)\left[\alpha\left(\frac{1}{n}\right) - \alpha\left(\frac{1}{n+1}\right)\right]\left(t - \frac{1}{n+1}\right) \\ &= t - d_n + (d_n - d_{n+1})[(n+1) - n(n+1)t] \\ &\ge t - \left[t - \psi(t)\right] + (d_n - d_{n+1})[(n+1) - n(n+1)t] \\ &\ge \psi(t). \end{aligned} \tag{1.4}$$

So $\psi(t) \le \alpha(t)$ for all $t > 0$.

Since $\alpha(n) < n$ and $\alpha(1/n) < 1/n$ for all $n \in \mathbb{N}$, so $\alpha(t) < t$ for all $t > 0$. □

Proposition 1.4. *If $\psi : \Re_+ \longrightarrow \Re_+$ is a generalized comparison function, then there exists a strictly increasing, continuous function $\alpha : \Re_+ \longrightarrow \Re_+$ such that*

$$\begin{gathered} \psi(t) \le \alpha(t) < t, \quad \text{for all } t > 0, \\ \lim_{t\to\infty} \alpha(t) = \infty. \end{gathered} \tag{1.5}$$

Proof. By Proposition 1.3, there exists a strictly increasing, continuous function $\overline{\alpha} : \Re_+ \longrightarrow \Re_+$ such that $\psi(t) \le \overline{\alpha}(t)$, for all $t > 0$. So, we may assume that $\lim_{t\to\infty}\alpha(t) = \infty$, by letting $\alpha(t) = (\overline{\alpha}(t) + t)/2$ for all $t \in \Re_+$. □

Remark 1.5. In the above case, the function α is invertible. If for each $t > 0$, we let $\alpha^0(t) = t$ and $\alpha^{-n}(t) = \alpha^{-1}(\alpha^{-n+1}(t))$ for all $n \in \mathbb{N}$, then we have that $\lim_{n\to\infty}\alpha^{-n}(t) = \infty$; that is, $\lim_{n\to\infty}\alpha^{n}(t) = 0$.

Proof. We claim that $\lim_{n\to\infty}\alpha^{n}(t) = 0$, for $t > 0$. Suppose that $\lim_{n\to\infty}\alpha^{-n}(t) = \eta$ for some positive real number η. Then

$$\eta = \lim_{n\to\infty}\alpha^{-n}(t) = \alpha^{-1}\left(\lim_{n\to\infty}\alpha^{-n+1}(t)\right) = \alpha^{-1}(\eta) > \eta, \tag{1.6}$$

which is a contradiction. So $\lim_{n\to\infty}\alpha^{n}(t) = 0$. □

We now are going to give the axiomatic definition for the measure of noncompactness in a complete metric space.

Definition 1.6. Let (M, d) be a metric space, and let $B(M)$ the family of bounded subsets of M. A map

$$\Phi : B(M) \longrightarrow [0,\infty) \tag{1.7}$$

is called a measure of noncompactness defined on M if it satisfies the following properties:

(i) $\Phi(D_1) = 0$ if and only if D_1 is precompact, for each $D_1 \in B(M)$,

(ii) $\Phi(\overline{D_1}) = \Phi(D_1)$, for each $D_1 \in B(M)$,

(iii) $\Phi(D_1 \cup D_2) = \max\{\Phi(D_1), \Phi(D_2)\}$, for each $D_1, D_2 \in B(M)$,

(iv) $\Phi(D_1) = \Phi(co(D_1))$, for each $D_1 \in B(M)$.

The above notion is a generalization of the set measure of noncompactness in metric spaces. The following α-measure is a well-known measure of noncompactness.

Definition 1.7. Let (M, d) be a complete metric space, and let $B(M)$ the family of bounded subsets of M. For each $D \in B(M)$, we define the set measure of noncompactness $\alpha(D)$ by:

$$\alpha(D) = \inf\left\{\varepsilon > 0 : \ D \text{ can be covered by finitely many sets with diameter} \leqslant \varepsilon\right\}. \tag{1.8}$$

Definition 1.8. Let X be a nonempty subset of a metric space (M, d). If a mapping $T : X \longrightarrow 2^M$ with for each $A \subset X$, A and $T(A)$ are bounded, then T is called

(i) a k-set contraction, if for each $A \subset X$, $\alpha(T(A)) \leq k\alpha(A)$, where $k \in [0, 1)$,

(ii) a weaker Meir-Keeler type ψ-set contraction, if for each $A \subset X$, $\alpha(T(A)) \leq \psi(\alpha(A))$, where $\psi : \mathfrak{R}_+ \longrightarrow \mathfrak{R}_+$ is a weaker Meir-Keeler type function,

(iii) a generalized comparison (comparison) type ψ-set contraction, if for each $A \subset X$, $\alpha(T(A)) \leq \psi(\alpha(A))$, where $\psi : \mathfrak{R}_+ \longrightarrow \mathfrak{R}_+$ is a generalized comparison (comparison) function.

Remark 1.9. It is clear that if $T : X \longrightarrow 2^M$ is a k-set contraction, then T is a weaker Meir-Keeler type ψ-set contraction, but the converse does not hold.

2. Main Results

Using the conception of the weaker Meir-Keeler type function, we establish the following important theorem.

Theorem 2.1. *Let X be a nonempty bounded subadmissible subset of a metric space (M, d). If $T : X \to 2^X$ is a weaker Meir-Keeler type ψ-set contraction with for each $t \in \Re_+$, $\{\psi^n(t)\}_{n\in\mathbb{N}}$ is nonicreasing, then X contains a precompact subadmissible subset K with $T(K) \subset K$.*

Proof. Take $y \in X$, and let

$$X_0 = X, \qquad X_1 = co(T(X_0) \cup \{y\}),$$
$$X_{n+1} = co(T(X_n) \cup \{y\}), \quad \text{for each } n \in N. \tag{2.1}$$

Then

(1) X_n is a subadmissible subset of X, for each $n \in N$;

(2) $T(X_n) \subset X_{n+1} \subset X_n$, for each $n \in N$.

Since $T : X \to 2^X$ is a weaker Meir-Keeler type ψ-set contraction, then $\alpha(T(X_n)) \leq \psi(\alpha(X_n))$ and $\alpha(X_{n+1}) = \alpha(co(T(X_n) \cup \{y\})) \leq \alpha(T(X_n))$. Hence, we conclude that $\alpha(X_n) \leq \psi^n(\alpha(X))$.

Since $\{\psi^n(\alpha(X))\}_{n\in\mathbb{N}}$ is nonincreasing, it must converge to some η with $\eta \geq 0$; that is, $\lim_{n\to\infty}\psi^n(\alpha(X)) = \eta \geq 0$. We now claim that $\eta = 0$. On the contrary, assume that $\eta > 0$.Then by the definition of the weaker Meir-Keeler type function, there exists $\delta > 0$ such that for each $A \subset X$ with $\eta \leq \alpha(A) < \eta + \delta$, there exists $n_0 \in \mathbb{N}$such that $\psi^{n_0}(\alpha(A)) < \eta$.Since $\lim_{n\to\infty}\psi^n(\alpha(X)) = \eta$, there exists $m_0 \in \mathbb{N}$ such that $\eta \leq \psi^m(\alpha(X)) < \eta + \delta$, for all $m \geq m_0$. Thus, we conclude that $\psi^{m_0+n_0}(\alpha(X)) < \eta$. So we get a contradiction. So $\lim_{n\to\infty}\psi^n(\alpha(X)) = 0$, and so $\lim_{n\to\infty}\alpha(X_n) = 0$.

Let $X_\infty = \cap_{n\in\mathbb{N}} X_n$. Then X_∞ is a nonempty precompact subadmissible subset of X, and by (2), we have $T(X_\infty) \subset X_\infty$. □

Remark 2.2. In the process of the proof of Theorem 2.1, we call the set X_∞ a Meir-Keeler type precompact-inducing subadmissible subset of X.

Applying Proposition 1.3, 1.4, and Remark 1.5, we are easy to conclude the following corollary.

Corollary 2.3. *Let X be a nonempty bounded subadmissible subset of a metric space (M, d). If $T : X \to 2^X$ is a generalized comparison (comparison) type ψ-set contraction, then X contains a precompact subadmissible subset K with $T(K) \subset K$.*

Proof. The proof is similar to the proof of Theorem 2.1; we omit it. □

Remark 2.4. In the process of the proof of Corollary 2.3, we also call the set X_∞ a generalized comparison type precompact-inducing subadmissible subset of X.

Corollary 2.5. *Let X be a nonempty bounded subadmissible subset of a metric space (M, d). If $T : X \to 2^X$ is a k-set contraction, then X contains a precompact subadmissible subset K with $T(K) \subset K$.*

Following the concepts of the KKM(X, Y) family (see [3]), we immediately have the following Lemma 2.6.

Lemma 2.6. *Let X be a nonempty subadmissible subset of a metric space (M, d), and let Y a topological spaces. Then $T|_D \in KKM(D, Y)$, whenever $T \in KKM(X, Y)$, and D is a nonempty subadmissible subset of X.*

We now concern a fixed point theorem for a weaker Meir-Keeler type ψ-set contraction in a complete metric space, which needs not to be a compact map.

Theorem 2.7. *Let X be a nonempty bounded subadmissible subset of a metric space (M, d). If $T \in KKM(X, X)$ is a weaker Meir-Keeler type ψ-set contraction with for each $t \in \Re_+$, $\{\psi^n(t)\}_{n\in\mathbb{N}}$ is nonicreasing, and closed with $\overline{T(X)} \subset X$, then T has a fixed point in X.*

Proof. By the same process of Theorem 2.1, we get a weaker Meir-Keeler type precompact-inducing subadmissible subset X_∞ of X. Since $\overline{T(X)} \subset X$ and $T(X_{n+1}) \subset T(X_n) \subset T(X)$ for each $n \in N$, we have $\overline{T(X_{n+1})} \subset \overline{T(X_n)} \subset X$ for each $n \in N$. Since $\alpha(\overline{T(X_n)}) \to 0$ as $n \to \infty$, by the above Lemma 2.6, we have that $\overline{T(X_\infty)}$ is a nonempty compact subset of X.

Since $T \in KKM(X, X)$ and X_∞ is a nonempty subadmissible subset of X, by Lemma 2.6, $T|_{X_\infty} \in KKM(X_\infty, X)$.

We now claim that for each ε, there exists an $x_\varepsilon \in X_\infty$ such that $B(x_\varepsilon, \varepsilon) \cap T(x_\varepsilon) \neq \phi$. If the above statement is not true, then there exists ε' such that $B(x, \varepsilon') \cap T(x) = \phi$, for all $x \in X_\infty$. Let $K = \overline{T(X_\infty)} \subset X$. Then we now define $F : X_\infty \to 2^K$ by

$$F(x) = K \setminus N(x, \varepsilon'), \quad \text{for each } x \in X_\infty. \tag{2.2}$$

Then

(1) $F(x)$ is compact, for each $x \in X_\infty$, and

(2) F is a generalized KKM mapping with respect to $T|_{X_\infty}$.

We prove (2) by contradiction. Suppose F is not a generalized KKM mapping with respect to $T|_{X_\infty}$. Then there exists $A = \{x_1, x_2, \ldots, x_n\} \in \langle X_\infty \rangle$ such that

$$T(co\{x_1, x_2, \ldots, x_n\}) \not\subseteq \cup_{i=1}^n F(x_i). \tag{2.3}$$

Choose $\mu \in co\{x_1, x_2, \ldots, x_n\}$ and $\nu \in T(\mu) \subset \overline{T(X_\infty)} = K$ such that $\nu \notin \cup_{i=1}^n F(x_i)$. From the definition of F, it follows that $\nu \in N(x_i, \varepsilon')$, for each $i \in \{1, 2, \ldots, n\}$. Since $\mu \in co\{x_1, x_2, \ldots, x_n\}$, $\nu \in T(\mu)$, we have $\mu \in co(A) \subset B(\nu, \varepsilon')$, which implies that $\nu \in B(\mu, \varepsilon')$. Therefore, $\nu \in T(\mu) \cap B(\mu, \varepsilon')$. This contradicts to $T(\mu) \cap B(\mu, \varepsilon') = \phi$. Hence, F is a generalized KKM mapping with respect to $T|_{X_\infty}$.

Since $T|_{X_\infty} \in$ KKM(X_∞, X), the family $\{F(x) : x \in X_\infty\}$ has the finite intersection property, and so we conclude that $\cap_{x\in X_\infty} F(x) \neq \phi$. Choose $\eta \in \cap_{x\in X_\infty} F(x)$, then $\eta \in K \setminus N(x, \varepsilon')$ for all $x \in X_\infty$. But, since $\eta \in \cap_{x\in X_\infty} F(x)$ and $K \subset \overline{X_\infty} \subset \cup_{x\in\overline{X_\infty}} N(x, (1/2)\varepsilon')$, so there exists an $x_0 \in X_\infty$ such that $\eta \in N(x_0, \varepsilon')$. So, we have reached a contradiction.

Therefore, we have proved that for each $\varepsilon > 0$, there exists an $x_\varepsilon \in X_\infty$ such that $B(x_\varepsilon, \varepsilon) \cap T(x_\varepsilon) \neq \phi$. Let $y_\varepsilon \in B(x_\varepsilon, \varepsilon) \cap T(\varepsilon)$. Since $y_\varepsilon \subset K$ and K is compact, we may assume that $\{y_\varepsilon\}$ converges to some $\overline{y} \in K$, then x_ε also converges to $\overline{y}$. Since T is closed, we have $\overline{y} \in T(\overline{y})$. This completes the proof. □

Corollary 2.8. *Let X be a nonempty bounded subadmissible subset of a metric space (M, d). If $T \in KKM(X, X)$ is a generalized composion type φ-set contraction and closed with $\overline{T(X)} \subset X$, then T has a fixed point in X.*

Corollary 2.9. *Let X be a nonempty bounded subadmissible subset of a metric space (M, d). If $T \in KKM(X, X)$ is a k-set contraction and closed with $\overline{T(X)} \subset X$, then T has a fixed point in X.*

The Φ-spaces, in an abstract convex space setting, were introduced by Amini et al.[7]. An abstract convex space $(X, \mathcal{C})$ consists of a nonempty topological space X and a family $\mathcal{C}$ of subsets of X such that X and ϕ belong to $\mathcal{C}$, and $\mathcal{C}$ is closed under arbitrary intersection. Let $(X, \mathcal{C})$ be an abstract convex space, and let Y a topological space. A map $T : Y \to 2^X$ is called a Φ-mapping if there exists a multifunction $F : Y \to 2^X$ such that

(i) for each $y \in Y$, $A \in \langle F(y) \rangle$ implies $ad_{\mathcal{C}}(A) \subset T(y)$;

(ii) $Y = \cup_{x \in X} \mathrm{int} F^{-1}(x)$.

The mapping F is called a companion mapping of T. Furthermore, if the abstract convex space $(X, \mathcal{C})$ which has a uniformity $\mathcal{U}$ and $\mathcal{U}$ has an open symmetric base family $\mathbb{N}$, then X is called a Φ-space if for each entourage $V \in \mathbb{N}$, there exists a Φ-mapping $T : X \to 2^X$ such that $\mathcal{G}_T \subset V$. Following the conceptions of the abstract convex space and the Φ-space, we are easy to know that a bounded metric space M is an important example of the abstract convex space, and if $X_1 \subset X$ and $\mathcal{C}_1 = \{C \cap X_1 : C \in \mathcal{C}\}$, then $(X_1, \mathcal{C}_1)$ is also a Φ-space.

Applying Theorem 2.5 of Amini et al. [7], we can deduce the following theorem in metric spaces.

Theorem 2.10. *Let X be a nonempty subadmissible subset of a metric space (M, d). If $T \in KKM(X, X)$ is compact, then for each $r > 0$, there exists $x_r \in X$; such that $B(x_r, r) \cap T(x_r) \neq \phi$.*

Proof. Consider the family $\mathcal{C}$ of all subadmissible subsets of M and for each $r > 0$, $x \in X$, we set $V_r[x] = B(x, r)$. Let

$$\mathbb{N} = \{V_r \mid V_r = \cup_{x \in M}\{(x, y) : y \in V_r[x],\ r > 0\}\}. \tag{2.4}$$

Then $\mathbb{N}$ is a basis of a uniformity of X. For each $V_r \in \mathbb{N}$, we define two set-valued mappings $G, F : X \to 2^X$ by $G(x) = T(x) = V_r[x]$ for each $x \in X$. Then we have

(i) for each $x \in X$, $ad_{\mathcal{C}}(G(x)) = ad_{\mathcal{C}}(V_r[x]) = V_r[x] = T(x) \subset V_r[T(x)]$;

(ii) $X = \cup_{x \in X} \mathrm{int} G^{-1}(x)$.

So, G is a companion mapping of F. This implies that F is a Φ-mapping such that $\mathcal{G}_F \subset V_r$. Therefore, $(X, \mathcal{C})$ is a Φ-space.

Now we let $s : X \to X$ be an identity mapping, all of the the conditions of Theorem 2.5 of Amini et al. [7] are fulfilled, and we can obtain the results. □

Applying Theorems 2.1 and 2.10, we can conclude the following fixed point theorems.

Theorem 2.11. *Let X be a nonempty bounded subadmissible subset of a metric space (M,d). If $T \in KKM(X,X)$ is a weaker Meir-Keeler type ψ-set contraction with for each $t \in \Re_+$, $\{\psi^n(t)\}_{n\in\mathbb{N}}$ is noincreasing, and closed with $\overline{T(X)} \subset X$, then T has a fixed point in X.*

Theorem 2.12. *Let X be a nonempty bounded subadmissible subset of a metric space (M,d). If $T \in KKM(X,X)$ is a generalized comparison (comparison) type ψ-set contraction and closed with $\overline{T(X)} \subset X$, then T has a fixed point in X.*

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